Crawling in crowed conditions. Application to network reconstruction.

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1 Introduction

Diffusion is one of most studied problem in nature. It investigates the transport phenomena due the spontaneous spreading of mass in a defined domain. The first empirical description of this process was formulated as the Fick's law of diffusion [1]. Successively, thanks first to Einstein [2] and then to Smoluchowski [3], the diffusion process was analytically modelled on the basis of previous brownian motion observations [4].

From then on, diffusion-like processes were applied to many areas, from physics to biology [5–7]. Moreover in the last years, attention has been focused on the dynamics of particles on specific discrete mediums, namely the complex networks. This interest rises from the similarity between the topological properties of these structures and those characterizing real life phenomena, such as cell compartments or traffic flow [8]. Networks have nowadays a central role in the modelling of physical problems, and thus the diffusion on such environments has become an emergent field of investigation [9–13]. Most of the studies done in this area focussed on the detection of community structures [10, 11, 14], or on the prediction of stationary patterns [15], and finally on the problem of diffusive epidemic process in a crowded environment [16].

2 Model

In all the previously presented models, it has been hypothesized the existence of a single walker or independent ones if many were active. In this work, we make one step forward by studying the simultaneous diffusion of several random walkers on a complex symmetric network in a crowded regime. More precisely, each individual crawler sitting on a node obeys the standard rules of random walk, jumps to distance 1 nodes are equiprobable, however each node can account for only a finite number of walkers (N), namely it possesses a finite carrying capacity. This implies that the transition probability from one node to another takes also into consideration the quantity of free available volume in the destination nodes.

Starting from this microscopic formulation, we derive a diffusion equation characterized by a transport operator that differs from the standard random-walk Laplacian matrix, notably for the presence of nonlinear terms involving products of the density of walkers:

$$\frac{\partial}{\partial t}\rho_i = \sum_{j=1}^{\Omega} \Delta_{ij} \left[\rho_j (1-\rho_i) - \frac{k_j}{k_i}\rho_i (1-\rho_j)\right] \tag{1}$$

where A_{ij} is the $\Omega \times \Omega$ symmetric adjacency matrix encoding the connections in the network, $k_i = \sum_j A_{ij}$ the degree of the i–th node, $\Delta_{ij} = A_{ij}/k_j - \delta_{ij}$ the random-walk normalized Laplacian matrix and the continuous variable $\rho_i(t)$ represents the concentration of particles at node i and it is related to the discrete variable n_i (number of walkers at node i) through $\rho_i = \lim_{N\to\infty}\langle n_i\rangle/N$.

The different structure of the diffusion is directly reflected on the stationary solution: the asymptotic concentration of crawlers in each node is no longer proportional to the degree of the node itself but it is given by a nonlinear function of the degree. We are able to derive an analytical formula for such solution valid for any crowding conditions and which returns the classical random walk distribution in the case of diluted systems, namely once the number of crawlers is very small with respect to the available volume:

$$\rho_i^{\infty} = \frac{ak_i}{1+ak_i} \quad \forall i = 1,\dots,\Omega\,, \tag{2}$$

where a is a parameter to be determined to satisfy the mass conservation constraint, $\sum_i \rho_i(t) = \sum_i \rho_i(0) = M$, which straightforwardly follows from Eq. (1).

3 Results

We conclude by presenting a main application of the previous theory devoted to the reconstruction of the unknown network topology upon which the crawlers move, remarkably enough a *single node measurement* is sufficient to achieve the goal. We are indeed able to reconstruct the degree distribution $p(k)$ using the information on $\rho_i(t)$ (with t large enough) observed on a single node and repeating s independent experiments involving different numbers of crawlers (whatever the crowding conditions).

More precisely selecting a node as starting point for all the walkers, say node $i = 1$, we can use Eq. (2) and get $a(M) = \rho_1^{\infty}/(1-\rho_1^{\infty}) \times 1/k_1$, where we emphasised the dependence on the *system mass* M, i.e. the total number of walker. Let us observe that $a(M)$ depends only on local measurable quantities: the node degree k_1 and the stationary distribution of walkers on the node, ρ_1^{∞}, that one can safely assume to be know if one waits long enough observing the number of walkers contained in node 1.

Introducing the number of nodes with degree k, $n(k)$, we can obtain from Eq. (2)

$$M = \sum_k n(k)\frac{a(M)k}{1+a(M)k}\,, \tag{3}$$

performing several *experiments*, namely random walks with a different number of walkers M_i, $i = 1, \dots, s$, one can rewrite the previous relation in the following form:

$$\begin{pmatrix} M_1 \\ \vdots \\ M_s \end{pmatrix} = F \begin{pmatrix} n(1) \\ \vdots \\ n(k_{max}) \end{pmatrix}, \tag{4}$$

where we introduced the matrix $F_{ij} = \frac{a_i j}{1+a_i j}$ and we wrote for short $a_i = a(M_i)$, that we recall is a known quantity.

Solving this linear system for the unknown $n(1), \dots, n(k_{max})$ we can reconstruct the degree distribution of the network. We successfully tested our method in both synthetic networks (see Fig. 1 for the case of an Erdős-Rény network and a Scale Free one) but also in realistic ones (*C. Elegans* neural network and the *karate club* network, data not shown).

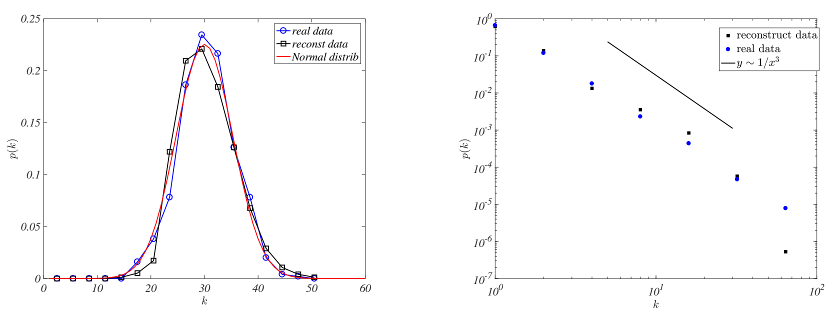

Fig. 1. Network reconstruction. Left panel: the degree distribution $p(k)$ for an Erdős-Rény random network made by $\Omega = 500$ nodes and probability to have a link between two nodes $p = 0.06$. Right panel: The degree distribution $p(k)$ for a Scale Free network made by $\Omega = 2000$ nodes and $\gamma = 3$ built using the Barabási-Albert algorithm. In both panels the blue circles denote the real probability distribution (i.e. computed from the knowledge of the network) while the black squares represent the reconstructed $p(k)$, the red line (left panel) is the asymptotic binomial distribution with parameters Ω and p, the black line (right panel) is the theoretical distribution $y \sim 1/x^3$.

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